

Novel symmetries in an interacting $\mathcal{N} = 2$ supersymmetric quantum mechanical model

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Abstract: We demonstrate the existence of a set of novel discrete symmetry transformations in the case of an interacting $\mathcal{N} = 2$ supersymmetric quantum mechanical model of a system of an electron moving on a sphere in the background of a magnetic monopole and establish its interpretation in the language of differential geometry. These discrete symmetries are, over and above, the usual *three* continuous symmetries of the theory which *together* provide the physical realizations of the de Rham cohomological operators of differential geometry. We derive the nilpotent $\mathcal{N} = 2$ SUSY transformations by exploiting our idea of supervariable approach and provide geometrical meaning to these transformations in the language of Grassmannian translational generators on a (1, 2)-dimensional supermanifold on which our $\mathcal{N} = 2$ SUSY quantum mechanical model is generalized. We express the conserved supercharges and the invariance of the Lagrangian in terms of the supervariables (obtained after the imposition of the SUSY invariant restrictions) and provide the geometrical meaning to (i) the nilpotency property of the $\mathcal{N} = 2$ supercharges, and (ii) the SUSY invariance of the Lagrangian of our $\mathcal{N} = 2$ SUSY theory.

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1 Introduction

It is a well-known fact that three (out of four) fundamental interactions of nature are governed by the gauge theories which are endowed with the first-class constraints in the language of Dirac's prescription for the classification scheme. These theories are characterized by the existence of local gauge symmetries which are generated by the above first-class constraints. There is a class of gauge theories that respect the dual-gauge symmetry transformations in addition to the above cited local gauge symmetry transformations. Such gauge theories provide the physical models for the Hodge theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism where the local gauge symmetries are traded with the nilpotent (anti-)BRST symmetries and the dual-gauge symmetry transformations are elevated to the (anti-) co-BRST symmetry transformations at the *quantum* level. In an earlier article (see, e.g. [1] and references therein), we have shown that such examples of gauge theories are the Abelian p -form ($p = 1, 2, 3$) gauge theories that are described within the framework of BRST formalism in $D = 2p$ dimensions of spacetime.

In a recent set of papers [2-5], we have established that the $\mathcal{N} = 2$ supersymmetric (SUSY) quantum mechanical models (QMMs) *also* provide a set of tractable physical examples of Hodge theory because their continuous symmetries (and corresponding conserved charges) provide the physical realizations of the de Rham cohomological operators* of differential geometry [6-10] and discrete symmetry transformations correspond to the physical analogue of the Hodge duality operation. We have also established the exact similarities between the Hodge algebra obeyed by the cohomological operators and algebra respected by the conserved SUSY charges and the Hamiltonian of the theory (which is nothing but one of the simplest $\mathcal{N} = 2$ SUSY algebra (i.e. $sl(1/1)$) without any central extension).

Such studies are important *physically* because exploiting the inputs from these kind of studies, we have been able to establish [11] that the two $(1 + 1)$ -dimensional (2D) *free* (non-)Abelian 1-form gauge theories (without any interaction with matter fields) are the *new* models for the topological field theories (TFTs) which capture some salient features of Witten-type TFTs and a few key features of Schwarz-type TFTs (see, e.g. [11-13] for details). We have *also* shown that the *above* free gauge field theoretic models and the 2D $U(1)$ gauge theory interacting with the Dirac fields [14,15] are the *perfect* models for the Hodge theory within the framework of BRST formalism. Such a set of Hodge theoretic models have *also* been proven in the description of the 2D modified versions of the Abelian 1-form anomalous gauge theory as well as Proca theory (where the gauge and matter fields are present [16,17] along with the mass term for the gauge fields).

In a very recent paper [5], we have established that the free version of the widely-studied *interacting* $\mathcal{N} = 2$ SUSY QMM of a charged particle (i.e. an electron) constrained to move on a sphere, in the background of a magnetic monopole, is a model for the Hodge theory.

*On a compact manifold without a boundary, there exists a set of three differential operators (d, δ, Δ) which are called as the de Rham cohomological operators of differential geometry. The (co-) exterior derivatives (δ) d are nilpotent of order two and their anticommutator defines the Laplacian operator Δ . They follow the algebra: $d^2 = \delta^2 = 0, \Delta = (d + \delta)^2 = \{d, \delta\}, [\Delta, d] = [\Delta, \delta] = 0$. The (co-)exterior derivatives are connected with each-other by the relationship: $\delta = \pm * d *$ where the $(*)$ stands for the Hodge duality operation on the given compact manifold. The Laplacian operator behaves like the Casimir operator (see, e.g. [6-10]) for this algebra.

The purpose of our present paper is to demonstrate that, in addition to the free case, the *interacting* $\mathcal{N} = 2$ SUSY QMM of the above system *also* provides a tractable physical example of Hodge theory where the conserved Noether charges of this theory follow exactly the same algebraic structure as that of the de Rham cohomological operators of differential geometry. Furthermore, we apply the theoretical arsenal of supervariable approach[†] [18–21] to derive the nilpotent $\mathcal{N} = 2$ SUSY transformations of this theory and provide the geometrical interpretations for the nilpotency property of the $\mathcal{N} = 2$ SUSY transformations (and their *generators*) as well as the SUSY invariance of the Lagrangian of our theory.

In the context of the statements made on the method of supervariable approach to derive the $\mathcal{N} = 2$ SUSY symmetries, we would like to mention that we have to concentrate on the (anti-)chiral supervariables so that we could capture *only* the nilpotency property of the SUSY symmetries (and avoid the property of absolute anticommutativity). In our present endeavor (and its predecessor [5]) we have accomplished this goal and we have provided the geometrical basis for the nilpotency and SUSY invariance of the Lagrangian for our present theory. The choice of the (anti-)chiral supervariables should be contrasted with the superfield approach to BRST formalism (see, e.g. [22–27]) where the superfields[‡], defined on the $(D, 2)$ -dimensional supermanifold, are expanded along *all* the Grassmannian directions of the supermanifold so that one could capture the nilpotency as well as the anticommutativity properties of the (anti-)BRST symmetries for a given D -dimensional gauge theory which is generalized onto the above supermanifold.

The main motivating factors behind our present endeavor are as follows. First and foremost, it is important for us to prove that the *interacting* $\mathcal{N} = 2$ SUSY QMM of the motion of an electron on a sphere in the background of a magnetic monopole is *also* a model for the Hodge theory (as has been shown by us that its *free* version is a tractable model for the Hodge theory [5]). Second, the physical realization of the abstract mathematical de Rham cohomological operators in the language of discrete and continuous symmetry transformations of a physical theory is *interesting* in its *own right*. Third, the goal of proving various SUSY models (e.g. $\mathcal{N} = 2, 4, 8, \dots$) to be the models for Hodge theory is theoretically important as it might turn out to be useful in the study of *such* SUSY gauge theories (in various dimensions of spacetime) which are important, at the moment, because of their connection with the superstring theories. Finally, our present endeavor (and its predecessor [5]), are our modest steps towards our main goal of proving the interacting $\mathcal{N} = 4$ SUSY QMM to provide a tractable SUSY model for the Hodge theory (because the generalization of our present interacting $\mathcal{N} = 2$ SUSY QMM to its counterpart $\mathcal{N} = 4$ SUSY quantum theory has already been discussed in the literature [28]).

The contents of our present investigation are organized as follows. In Sec. 2, we recapitulate the bare essentials of the nilpotent $\mathcal{N} = 2$ SUSY transformations and show

[†]We christen our present approach as the supervariable approach because when we set the Grassmannian variables equal to zero in a supervariable (defined on the $(1, 2)$ -dimensional supermanifold), we obtain an ordinary variable which is a function of t only in the realm of SUSY quantum mechanics.

[‡]When we set the Grassmannian variables equal to zero in a superfield (defined on a $(D, 2)$ -dimensional supermanifold), we obtain an ordinary field which is a function of the D -dimensional spacetime ordinary coordinates. This is why, we christen the approach, adopted in the realm of BRST formalism, as the superfield approach because it is applied to the description of an ordinary D -dimensional gauge field theory.

that their anticommutator generates a bosonic symmetry in the theory. We compute the conserved charges by exploiting the basic concepts of Noether's theorem. Our Sec. 3 is devoted to the discussion of a set of novel discrete symmetry transformations which turn out to be responsible for establishing a useful connection between the two nilpotent $\mathcal{N} = 2$ SUSY transformations. In Sec. 4, we derive the algebraic structure of the symmetry transformations in their operator form and show the existence of one of the simplest form of $\mathcal{N} = 2$ SUSY algebra amongst the conserved charges. Our Sec. 5 deals with the connection between $\mathcal{N} = 2$ SUSY algebra and the de Rham cohomological operators of differential geometry thereby providing the proof that our present model is a SUSY quantum mechanical example of a Hodge theory. Sec. 6 deals with the derivation of $\mathcal{N} = 2$ SUSY transformations by exploiting the supervariable approach. Finally, we make some concluding remarks in our Sec. 7 and point out a few future directions for further investigation.

In our Appendix A, for the readers' convenience, we mention a few key points that are connected with the superspace formalism where the (anti-)chiral supervariables are used [29]. Our Appendices B, C and D are devoted to clarify some of the expressions/equations that have been used in the main body of our text.

2 Preliminaries: Lagrangian formulation

Let us begin with the following Lagrangian for the $\mathcal{N} = 2$ SUSY quantum mechanical model of the motion an electron on a sphere in the background of the Dirac's magnetic monopole based on the $CP^{(1)}$ -model approach (see, e.g. [29] for details)

$$L = 2(D_t \bar{z}) \cdot (D_t z) + \frac{i}{2} [\bar{\psi} \cdot (D_t \psi) - (D_t \bar{\psi}) \cdot \psi] - 2ga, \quad (1)$$

where t is the evolution parameter in the theory and $\partial_t = \partial/\partial t$. Here the dot product between \bar{z} and z is taken to be $\bar{z} \cdot z = \sum_{i=1}^2 |z_i|^2$ which clearly demonstrates that

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \bar{z} = (z_1^* \quad z_2^*) \implies \bar{z} \cdot z = |z_1|^2 + |z_2|^2. \quad (2)$$

Similarly, other dot products, defined in the Lagrangian (1), should be taken into account. We also have the ‘‘covariant’’ derivatives $D_t z = (\partial_t - ia)z$, $D_t \bar{z} = (\partial_t + ia)\bar{z}$, $D_t \psi = (\partial_t - ia)\psi$, $D_t \bar{\psi} = (\partial_t + ia)\bar{\psi}$ where a is the ‘‘gauge’’ variable and the complex variables z and \bar{z} are bosonic in nature while their superpartners ψ and $\bar{\psi}$ are fermionic (i.e. $\psi^2 = \bar{\psi}^2 = 0, \psi \bar{\psi} + \bar{\psi} \psi = 0$) at the classical level[§]. In the Lagrangian (1), the parameter g stands for the charge on the monopole which interacts with the electron via gauge variable a through the coupling $(-2ga)$. This is the last term in the Lagrangian (1). The charge on the electron has been set equal to (-1) and its mass has been taken to be $(+1)$.

[§]We lay emphasis on the fact that the absolute anticommutativity ($\psi \bar{\psi} + \bar{\psi} \psi = 0$) is true only at the *classical* level. As is evident from equation (9) (see, below), at the *quantum* level, ψ and $\bar{\psi}$ would *not* absolutely anticommute because they are canonically conjugate to each-other.

The above Lagrangian respects the following infinitesimal, continuous and nilpotent ($s_1^2 = s_2^2 = 0$) $\mathcal{N} = 2$ SUSY transformations (s_1, s_2):

$$\begin{aligned}
s_1 z &= \frac{\psi}{\sqrt{2}}, & s_1 \psi &= 0, & s_1 \bar{\psi} &= \frac{2i D_t \bar{z}}{\sqrt{2}}, & s_1 \bar{z} &= 0, \\
s_1 (D_t z) &= \frac{D_t \psi}{\sqrt{2}}, & s_1 (D_t \bar{z}) &= 0, & s_1 a &= 0, \\
s_2 \bar{z} &= \frac{\bar{\psi}}{\sqrt{2}}, & s_2 \bar{\psi} &= 0, & s_2 \psi &= \frac{2i D_t z}{\sqrt{2}}, & s_2 z &= 0, \\
s_2 (D_t \bar{z}) &= \frac{D_t \bar{\psi}}{\sqrt{2}}, & s_2 (D_t z) &= 0, & s_2 a &= 0,
\end{aligned} \tag{3}$$

because the Lagrangian (1) transforms to the total time derivatives as

$$s_1 L = \frac{d}{dt} \left[\frac{(D_t \bar{z}) \cdot \psi}{\sqrt{2}} \right], \quad s_2 L = \frac{d}{dt} \left[\frac{\bar{\psi} \cdot (D_t z)}{\sqrt{2}} \right]. \tag{4}$$

The above expression demonstrates that the $\mathcal{N} = 2$ SUSY transformations s_1 and s_2 are the *symmetry* transformations for the action integral $S = \int dt L$. It is good to mention that the “gauge” variable, defined in the following fashion[¶]

$$a = -\frac{i}{2} (\bar{z} \cdot \dot{z} - \dot{\bar{z}} \cdot z) - \frac{1}{2} (\bar{\psi} \cdot \psi), \tag{5}$$

remains invariant^{||} ($s_1 a = s_2 a = 0$) under the $\mathcal{N} = 2$ SUSY transformations s_1 and s_2 due to the supersphere defined by the following constraints (see, e.g. [29])

$$\bar{z} \cdot z - 1 = 0, \quad \bar{z} \cdot \psi = 0, \quad \bar{\psi} \cdot z = 0, \tag{6}$$

which define the $CP^{(1)}$ model on a sphere $\bar{z} \cdot z = 1$ (that is supersymmetrized with the inclusion of the above fermionic variables ψ and $\bar{\psi}$). It is worthwhile to mention that the invariance of the constraints ($\bar{z} \cdot z - 1 = 0$) under the $\mathcal{N} = 2$ SUSY transformations s_1 and s_2 , leads to the constraints $\bar{z} \cdot \psi = 0$ and $\bar{\psi} \cdot z = 0$. Mathematically, these constraints have also been determined by the superspace formalism in [29] which has been concisely mentioned in our Appendix A for the readers’ convenience.

We obtain a bosonic symmetry (s_ω) in the theory which is nothing but the anticommutator $s_\omega = \{s_1, s_2\}$ of the $\mathcal{N} = 2$ SUSY transformations s_1 and s_2 . The dynamical variables of our theory transform, under s_ω , as follows:

$$s_\omega z = D_t z, \quad s_\omega \bar{z} = D_t \bar{z}, \quad s_\omega \psi = D_t \psi, \quad s_\omega \bar{\psi} = D_t \bar{\psi}, \tag{7}$$

[¶]Actual superspace formalism [29] yields the expression for this “gauge” variable (see, Appendix A) as: $a = -([i(\bar{z} \cdot \dot{z} - \dot{\bar{z}} \cdot z) + (\bar{\psi} \cdot \psi)]/[2\bar{z} \cdot z])$. However, the substitution of the constraint $\bar{z} \cdot z = 1$ leads to the expression for a as quoted in (5) which avoids the presence of the variables in the denominator as well as singularity in our present theory.

^{||}The explicit application of s_1 and s_2 on a yields the results: $s_1 a = \frac{d}{dt} [-\frac{i}{2\sqrt{2}} (\bar{z} \cdot \psi)] + \frac{a}{\sqrt{2}} (\bar{z} \cdot \psi)$ and $s_2 a = \frac{d}{dt} [+ \frac{i}{2\sqrt{2}} (\bar{\psi} \cdot z)] + \frac{a}{\sqrt{2}} (\bar{\psi} \cdot z)$ which, finally, imply that $s_1 a = s_2 a = 0$ due to the constraint conditions: $\bar{z} \cdot \psi = 0$ and $\bar{\psi} \cdot z = 0$ which define the supersymmetrized version of the sphere ($\bar{z} \cdot z = 1$).

modulo a factor of i . Thus, it is clear that the anticommutator of two SUSY transformations generates the “covariant” time translation for the variables. This is one of the key requirements of a consistent $\mathcal{N} = 2$ interacting SUSY theory which is clear from the superspace and the (anti-)chiral supervariable formalism (see, Appendix A). Under the bosonic symmetry transformations (7), the Lagrangian transforms as: $s_\omega L = \frac{d}{dt}[L + 2ga] \equiv \frac{d}{dt}[2(D_t \bar{z}) \cdot (D_t z) + \frac{i}{2} \{\bar{\psi} \cdot (D_t \psi) - (D_t \bar{\psi}) \cdot \psi\}]$. As a consequence, the action integral of our theory remains invariant under s_ω for the physically well-defined variables.

The existence of continuous symmetries in a theory leads to the derivation of conserved Noether charges which turn out to be the generators for the continuous symmetry transformations. The continuous and nilpotent $\mathcal{N} = 2$ SUSY transformations s_1 and s_2 and their anticommutator s_ω lead to the derivation of the following Noether conserved charges

$$\begin{aligned}
Q &= \frac{\Pi_z \cdot \psi}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left[2 D_t \bar{z} + \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) \bar{z} + 2 i a \bar{z} (1 - \bar{z} \cdot z) \right] \cdot \psi \\
&\equiv \frac{1}{\sqrt{2}} \left[2 D_t \bar{z} + \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) \bar{z} \right] \cdot \psi, \\
\bar{Q} &= \frac{\bar{\psi} \cdot \Pi_{\bar{z}}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \bar{\psi} \cdot \left[2 D_t z - \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) z - 2 i a z (1 - \bar{z} \cdot z) \right] \\
&\equiv \frac{1}{\sqrt{2}} \bar{\psi} \cdot \left[2 D_t z - \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) z \right], \\
Q_\omega &= H = 2 (D_t \bar{z}) \cdot (D_t z) - \frac{1}{2} (\bar{\psi} \cdot \psi + 2g) (\bar{\psi} \cdot \psi) \\
&\equiv \frac{1}{2} \Pi_z \cdot \Pi_{\bar{z}} - \frac{1}{8} (\bar{\psi} \cdot \psi + 2g)^2,
\end{aligned} \tag{8}$$

where H is the Hamiltonian of our present theory. It is evident that, we have used the constraint $\bar{z} \cdot z = 1$ in the shorter version of Q and \bar{Q} in the above equation (8).

We note that the above charges have also been expressed in terms of canonical conjugate momenta Π_z and $\Pi_{\bar{z}}$ w.r.t. the dynamical variables z and \bar{z} . In fact, it is elementary to check that the canonical momenta (that emerge from Lagrangian (1)) corresponding to the dynamical variables z, \bar{z}, ψ and $\bar{\psi}$ are:

$$\begin{aligned}
\Pi_z &= \frac{\partial L}{\partial \dot{z}} = 2 D_t \bar{z} + \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) \bar{z} + 2 i a \bar{z} (1 - \bar{z} \cdot z), \\
&\equiv 2 D_t \bar{z} + \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) \bar{z}, \\
\Pi_{\bar{z}} &= \frac{\partial L}{\partial \dot{\bar{z}}} = 2 D_t z - \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) z - 2 i a z (1 - \bar{z} \cdot z), \\
&\equiv 2 D_t z - \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) z, \\
\Pi_\psi &= \frac{\partial L}{\partial \dot{\psi}} \equiv -\frac{i}{2} \bar{\psi}, \quad \Pi_{\bar{\psi}} = \frac{\partial L}{\partial \dot{\bar{\psi}}} \equiv -\frac{i}{2} \psi,
\end{aligned} \tag{9}$$

where we have adopted the convention of left derivative w.r.t. the fermionic superpartners ψ and $\bar{\psi}$ in the computation of Π_ψ and $\Pi_{\bar{\psi}}$. The equivalent forms of the canonical momenta $(\Pi_z, \Pi_{\bar{z}})$ have been derived by imposing the constraint $\bar{z} \cdot z = 1$. We also note

that the Hamiltonian H of our theory [cf. (8)] can also be computed by the Legendre transformations** $H = \Pi_z \cdot \dot{z} + \dot{\bar{z}} \cdot \Pi_{\bar{z}} - \Pi_\psi \cdot \dot{\psi} + \dot{\bar{\psi}} \cdot \Pi_{\bar{\psi}} - L$ by exploiting the definition of canonical conjugate momenta (9) and the expression for the Lagrangian (1). In this derivation, we have to use the constraint $\bar{z} \cdot z = 1$ and definitions of $D_t z$, $D_t \bar{z}$ and $a = -1/2[i(\bar{z} \cdot \dot{z} - \dot{\bar{z}} \cdot z) + \bar{\psi} \cdot \psi]$.

The Noether charges (Q, \bar{Q}, Q_ω) are conserved and their conservation law (i.e. $\dot{Q} = \dot{\bar{Q}} = \dot{Q}_\omega = 0$) can be proven by exploiting the following equations of motion (along with the constraint $\bar{z} \cdot z = 1$) that emerge from the Lagrangian (1) of our theory, namely;

$$\begin{aligned} \frac{d\Pi_{\bar{z}}}{dt} - i \left[2a D_t z + \frac{\dot{z}}{2} (\bar{\psi} \cdot \psi + 2g) \right] &= 0, \\ \frac{d\Pi_z}{dt} + i \left[2a D_t \bar{z} + \frac{\dot{\bar{z}}}{2} (\bar{\psi} \cdot \psi + 2g) \right] &= 0, \\ D_t \psi - \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) \psi &= 0, \quad D_t \bar{\psi} + \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) \bar{\psi} = 0, \end{aligned} \quad (10)$$

where the canonical conjugate momenta ($\Pi_z, \Pi_{\bar{z}}$) are defined in the equation (9). It is worth pointing out that the above charges are conserved on the constrained surface defined by the constraints $\bar{z} \cdot z = 1$, $\bar{z} \cdot \psi = 0$, $\bar{\psi} \cdot z = 0$. Thus, in the explicit proof of the conservation law, the EOM and constraints *both* are exploited together (see, Appendix B).

3 Novel discrete symmetry transformations

It is straightforward to note that the Lagrangian (1) remains invariant under the following discrete symmetry transformations, namely;

$$\begin{aligned} z &\rightarrow \mp \bar{z}, & \bar{z} &\rightarrow \mp z, & \psi &\rightarrow \mp \bar{\psi}, & \bar{\psi} &\rightarrow \pm \psi, \\ t &\rightarrow -t, & a &\rightarrow +a, & g &\rightarrow g, \end{aligned} \quad (11)$$

where we note that there is a time-reversal (i.e. $t \rightarrow -t$) symmetry. In other words, we observe that, in reality, the transformation $z \rightarrow \mp \bar{z}$ denotes that $z(t) \rightarrow z(-t) = \mp \bar{z}^T(t)$ where the superscript T on \bar{z} denotes the transpose operation on \bar{z} . We have suppressed the transpose operations in the transformation (11). This operation, however, should be taken into account in the rest of the discrete symmetry transformations (e.g. $\psi \rightarrow \mp \bar{\psi} \Rightarrow \psi(t) \rightarrow \psi(-t) = \mp \bar{\psi}^T(t)$, $\bar{\psi} \rightarrow \psi \Rightarrow \bar{\psi}(t) \rightarrow \bar{\psi}(-t) = \pm \psi^T(t)$, $a(t) \rightarrow a(-t) = a(t)$, etc.).

The above discrete symmetry transformations are useful and important because they facilitates a connection between the *two* nilpotent ($s_1^2 = s_2^2 = 0$) SUSY transformations s_1 and s_2 in the following manner, namely;

$$s_2 = \pm * s_1 *, \quad (12)$$

where the notation $*$ has been exploited for the discrete symmetry transformations (11). The (\pm) signs in the above equation are dictated by two successive operations of the discrete

** We note that the *ordering* and the proper *signatures* have been taken into account in our definition of the Legendre transformation which leads to the derivation of the Hamiltonian H .

symmetry transformations, namely;

$$\begin{aligned} * (* \Phi_1) &= + \Phi_1, & \Phi_1 &= z, \bar{z}, \\ * (* \Phi_2) &= - \Phi_2, & \Phi_2 &= \psi, \bar{\psi}. \end{aligned} \quad (13)$$

As a side remark, we observe that the bosonic variables $z(t)$ and $\bar{z}(t)$ have the (+) sign when they are acted upon by two successive discrete symmetry transformations. On the contrary, the fermionic variables (i.e. $\psi(t)$ and $\bar{\psi}(t)$) acquire a (-) sign under the above transformations. Thus, we have the following relationships:

$$\begin{aligned} s_2 \Phi_1 &= + * s_1 * \Phi_1 \Rightarrow s_2 = + * s_1 *, & \Phi_1 &= z, \bar{z}, \\ s_2 \Phi_2 &= - * s_1 * \Phi_2 \Rightarrow s_2 = - * s_1 *, & \Phi_2 &= \psi, \bar{\psi}, \end{aligned} \quad (14)$$

where there is an elegant interplay between the discrete and continuous symmetries of our theory. We note that there is existence of a set of reciprocal relationships, namely;

$$\begin{aligned} s_1 \Phi_1 &= - * s_2 * \Phi_1 \Rightarrow s_1 = - * s_2 *, & \Phi_1 &= z, \bar{z}, \\ s_1 \Phi_2 &= + * s_2 * \Phi_2 \Rightarrow s_1 = + * s_2 *, & \Phi_2 &= \psi, \bar{\psi}, \end{aligned} \quad (15)$$

which is also true for a duality invariant theory [30].

We end this section with the remarks that the relationships (12) between the $\mathcal{N} = 2$ SUSY transformations s_1 and s_2 are reminiscent of the relationships that exist between the exterior and co-exterior derivatives of differential geometry (i.e. $\delta = \pm * d *$). Thus, it is very interesting to note that the relationship (12) provides a physical realization for the above mathematical relationship in the language of the interplay between discrete and continuous symmetries of our present theory. We would like to lay emphasis on the fact that we have discussed only one set of discrete symmetry transformation in (11). However, there might exist many other useful discrete symmetry transformations (see, Appendix D) in our theory, too. We shall focus, however, for our rest of the discussions on (11) as the discrete symmetry transformations of our theory. It will be noted that the conserved charges (Q, \bar{Q}, Q_ω) transform, under the discrete symmetry transformations (11), as:

$$* Q = -\bar{Q}, \quad * \bar{Q} = Q, \quad * H = H. \quad (16)$$

Thus, we point out that the transformations of the SUSY charges Q and \bar{Q} are exactly like the duality transformation of Maxwell's theory of electrodynamics where $\vec{E} \rightarrow \vec{B}$, $\vec{B} \rightarrow -\vec{E}$ (under the Maxwell duality symmetry transformations). Furthermore, we observe that a couple of successive operations of the discrete symmetry transformation on the charges yield the following transformations:

$$* (* Q) = -Q, \quad * (* \bar{Q}) = -\bar{Q}, \quad * (* H) = H. \quad (17)$$

The above observation establishes the fermionic nature of Q and \bar{Q} as we have seen earlier in the case of ψ and $\bar{\psi}$ (i.e. $* (* \psi) = -\psi$, $* (* \bar{\psi}) = -\bar{\psi}$). It is worthwhile to point out that the algebra: $Q^2 = \bar{Q}^2 = 0$, $\{Q, \bar{Q}\} = H$, $[H, Q] = [H, \bar{Q}] = 0$ remains invariant under a couple of successive discrete symmetry transformations.

4 Operators and their algebra

We focus on the algebraic structure that is followed by the transformation operators (s_1, s_2, s_ω) and their corresponding Noether charges (Q, \bar{Q}, Q_ω) . In this respect, as has already been discussed, we note that the following is true, namely;

$$\begin{aligned} s_1^2 &= 0, & s_2^2 &= 0, & \{s_1, s_2\} &= s_\omega \equiv (s_1 + s_2)^2, \\ [s_\omega, s_1] &= 0, & [s_\omega, s_2] &= 0, & \{s_1, s_2\} &\neq 0. \end{aligned} \quad (18)$$

Thus, we observe that the bosonic symmetry operator (s_ω) commutes with all the *three* continuous symmetry operators (s_1, s_2, s_ω) of our interacting $\mathcal{N} = 2$ SUSY QMM. Hence, it behaves like the Casimir operator. In exactly similar fashion, it can be seen that (due to the relationship between the continuous symmetries and generators) we have the following:

$$\begin{aligned} s_1 Q &= i \{Q, Q\} = 0, & s_2 \bar{Q} &= i \{\bar{Q}, \bar{Q}\} = 0, \\ s_1 \bar{Q} &= i \{\bar{Q}, Q\} = i H, & s_2 Q &= i \{Q, \bar{Q}\} = i H, \\ s_\omega Q &= -i [Q, H] = 0, & s_\omega \bar{Q} &= -i [\bar{Q}, H] = 0. \end{aligned} \quad (19)$$

The above relationships lead to the realization of one of the simplest form of the $\mathcal{N} = 2$ SUSY algebra amongst the generators of (s_1, s_2, s_ω) , namely;

$$\begin{aligned} Q^2 &= \frac{1}{2} \{Q, Q\} = 0, & \bar{Q}^2 &= \frac{1}{2} \{\bar{Q}, \bar{Q}\} = 0, \\ \{Q, \bar{Q}\} &= H, & [H, Q] &= [H, \bar{Q}] = 0, \end{aligned} \quad (20)$$

where there is *no* central extension. Thus, the continuous symmetry transformations and their generators respect the celebrated $sl(1/1)$ algebra (without any central extension). The proof of (19) is algebraically a bit involved. Thus, for the readers' convenience, we have explicitly derived these relations in our Appendix C.

Physically, the above algebra can be understood as follows. The nilpotency ($Q^2 = 0$, $\bar{Q}^2 = 0$) of the charges (Q, \bar{Q}) shows that these charges are fermionic in nature. The vanishing of the commutators (i.e. $[Q, H] = [\bar{Q}, H] = 0$) demonstrates that the SUSY (i.e. fermionic) charges (Q, \bar{Q}) are conserved. Finally, the algebraic structure $\{Q, \bar{Q}\} = H$ says that the operation of two consecutive operations of s_1 and s_2 on any arbitrary variable lead to the “covariant” time translation [cf. Eqn. (7) of Sec. 2] of the *same* variable. This physical statement is mathematically captured by the presence of bosonic symmetry transformation (s_ω) in our theory (i.e. $s_\omega = \{s_1, s_2\}$).

The algebraic structures (18) and (20) are reminiscent of the algebra obeyed by the de Rham cohomological operators of differential geometry, namely;

$$d^2 = 0, \quad \delta^2 = 0, \quad \Delta = (d + \delta)^2 = \{d, \delta\}, \quad [\Delta, d] = 0, \quad [\Delta, \delta] = 0, \quad (21)$$

where the operators (d, δ, Δ) form a set called the de Rham cohomological operators of differential geometry. Thus, we note that the Laplacian operator Δ behaves like a Casimir operator^{††} exactly as s_ω and H behave in our equations (18) and (20). We note that the

^{††}This Laplacian operator is not like the precise Casimir operator of the Lie algebra (as the Hodge algebra (21) is *not* a Lie algebra). Similarly, the algebras (18) and (20) are *also* not Lie algebra.

continuous symmetry operators (s_1, s_2, s_ω) and their generators (Q, \bar{Q}, Q_ω) provide the physical realization of the cohomological operators (d, δ, Δ) of differential geometry in the language of symmetries and conserved charges.

We close this section with the remark that the well-known relation: $\delta = \pm * d *$ (that exists between the co-exterior derivative and exterior derivative) is realized in the language of the interplay between the discrete and continuous symmetries of our present theory [cf. (12)]. Thus, as far as the algebraic structure is concerned, we note that there is complete similarity amongst the relations (18), (20) and (21) and there is a mapping between (d, δ, Δ) and the conserved charges (Q, \bar{Q}, Q_ω) . This identification is, however, still not complete because there are cohomological properties that are associated with (d, δ, Δ) and we have to capture these properties in the language of conserved charges (Q, \bar{Q}, Q_ω) . We shall dwell on these aspects in our next section.

5 Cohomological connections

It is a well-known fact that when the de Rham cohomological operators (d, δ, Δ) act on a given form (i.e. $f^{(p)}$) of degree p , the following consequences ensue. First, the degree of this form increases by one when it is acted upon by the exterior derivative d (i.e. $d f^{(p)} \sim f^{(p+1)}$). On the contrary, the operation of δ on the same form (i.e. $f^{(p)}$) lowers the degree of the form by one (i.e. $\delta f^{(p)} \sim f^{(p-1)}$). It turns out that the operation of the Laplacian operator does *not* change the degree of the form at all (i.e. $\Delta f^{(p)} \sim f^{(p)}$). These properties are the essential ingredients when we discuss the cohomological aspects of a differential form w.r.t. the de Rham cohomological operators. We have to find the analogues of these observations in the language of the symmetry properties and conserved charges of the interacting $\mathcal{N} = 2$ SUSY QMM of our present endeavor and establish a proper relationship.

We have noted that H is like the Casimir operator for the $sl(1/1)$ algebra: $Q^2 = \bar{Q}^2 = 0$, $\{Q, \bar{Q}\} = H$, $[H, Q] = [H, \bar{Q}] = 0$. As a consequence of the latter two relations, it is evident that $H^{-1} Q = Q H^{-1}$ and $H^{-1} \bar{Q} = \bar{Q} H^{-1}$ (where we assume that H^{-1} is properly defined for the Hamiltonian of our theory). We note that the following relations turn out to be true due to the validity of the $sl(1/1)$ algebra, namely;

$$\begin{aligned} \left[\frac{Q \bar{Q}}{H}, Q \right] &= +Q, & \left[\frac{Q \bar{Q}}{H}, \bar{Q} \right] &= -\bar{Q}, \\ \left[\frac{\bar{Q} Q}{H}, Q \right] &= -Q, & \left[\frac{\bar{Q} Q}{H}, \bar{Q} \right] &= +\bar{Q}, \end{aligned} \quad (22)$$

where it has been taken for granted that $H^{-1} Q = Q H^{-1}$ and $H^{-1} \bar{Q} = \bar{Q} H^{-1}$ are very much true. Let us define now a state $|\chi\rangle_p$ (in the total quantum Hilbert space of states), with an eigenvalue equation w.r.t. the Hermitian operator $(Q \bar{Q}/H)$, as:

$$\left(\frac{Q \bar{Q}}{H} \right) |\chi\rangle_p = p |\chi\rangle_p, \quad (23)$$

where p is the eigenvalue which is *real* because $(Q \bar{Q}/H)$ operator is Hermitian. Now, we can discuss about the impact of the algebra (22) on the state $|\chi\rangle_p$ which could be useful

in the description of the cohomological aspects of states. In this context, we shall utilize the beauty of equation (22) as it behaves like the ladder operators of quantum mechanics. For instance, using the top two relationships from (22), we find that

$$\begin{aligned}\left(\frac{Q\bar{Q}}{H}\right) Q|\chi\rangle_p &= (p+1) Q|\chi\rangle_p, \\ \left(\frac{Q\bar{Q}}{H}\right) \bar{Q}|\chi\rangle_p &= (p-1) \bar{Q}|\chi\rangle_p, \\ \left(\frac{Q\bar{Q}}{H}\right) H|\chi\rangle_p &= p H|\chi\rangle_p.\end{aligned}\tag{24}$$

Thus, it is crystal clear that the states $Q|\chi\rangle_p$, $\bar{Q}|\chi\rangle_p$ and $H|\chi\rangle_p$ have the eigenvalues $(p+1)$, $(p-1)$ and p , respectively, w.r.t. the Hermitian operator $(Q\bar{Q}/H)$. These consequences are *exactly* same as the results that emerge from the operations of the cohomological operators (d, δ, Δ) on a form $f^{(p)}$ of degree p (because we do obtain the forms of degree $(p+1)$, $(p-1)$ and p , respectively) due to their operations.

Now we are in a position to exploit the mathematical beauty and power of the lower *two* entries of equation (22). If we define an eigenstate $|\xi\rangle_q$ (in the total quantum Hilbert space of states) w.r.t. the Hermitian operator $(\bar{Q}Q/H)$, as

$$\left(\frac{\bar{Q}Q}{H}\right) |\xi\rangle_q = q |\xi\rangle_q,\tag{25}$$

it is evident that the following consequences ensue, namely;

$$\begin{aligned}\left(\frac{\bar{Q}Q}{H}\right) \bar{Q}|\xi\rangle_q &= (q+1) \bar{Q}|\xi\rangle_q, \\ \left(\frac{\bar{Q}Q}{H}\right) Q|\xi\rangle_q &= (q-1) Q|\xi\rangle_q, \\ \left(\frac{\bar{Q}Q}{H}\right) H|\xi\rangle_q &= q H|\xi\rangle_q,\end{aligned}\tag{26}$$

which shows that the eigenvalues of states $\bar{Q}|\xi\rangle_q$, $Q|\xi\rangle_q$ and $H|\xi\rangle_q$ are $(q+1)$, $(q-1)$ and q , respectively, w.r.t. the Hermitian operator $(\bar{Q}Q/H)$. Once again, we note that these consequences are exactly same as the operations of the cohomological operators (d, δ, Δ) on the differential form of degree q . Hence, these mathematical operators can be realized in the language of conserved charges of our present theory and explained below.

Due to the beauty of equations (24) and (26), we observe that if an eigenstate $|\chi\rangle_p$ has an eigenvalue p w.r.t. the Hermitian operator $(Q\bar{Q}/H)$, the states $Q|\chi\rangle_p$, $\bar{Q}|\chi\rangle_p$ and $H|\chi\rangle_p$ would have the eigenvalues $(p+1)$, $(p-1)$ and p , respectively. Now, if we identify a corresponding differential form $f^{(p)}$ of degree p , we know that $df^{(p)}$, $\delta f^{(p)}$ and $\Delta f^{(p)}$ would have the degrees $(p+1)$, $(p-1)$ and p , respectively. Thus, we conclude that we have a mapping:

$$(Q, \bar{Q}, H) \iff (d, \delta, \Delta),\tag{27}$$

between the two kinds of operators. On the other hand, if we identify the eigenstate $|\xi\rangle_q$ with eigenvalue q , w.r.t. the Hermitian operator $(\bar{Q}Q/H)$, we observe that the quantum states $\bar{Q}|\xi\rangle_q$, $Q|\xi\rangle_q$ and $H|\chi\rangle_q$ would have their eigenvalues $(q+1)$, $(q-1)$ and q , respectively. Similarly, the forms $d f^{(q)}$, $\delta f^{(q)}$ and $\Delta f^{(q)}$ would carry the degrees $(q+1)$, $(q-1)$ and q , respectively, if we start with a form $f^{(q)}$ of degree q . Thus, we have the mapping of the operators:

$$(\bar{Q}, Q, H) \Longleftrightarrow (d, \delta, \Delta), \quad (28)$$

which are defined in two different spaces. Whereas, the operators (\bar{Q}, Q, H) are defined in the quantum Hilbert space of states, the operators (d, δ, Δ) are defined in the space of forms on a given manifold. All the above discussions imply that there are *two* realizations of (d, δ, Δ) in the language of conserved charges of our present $\mathcal{N} = 2$ SUSY theory. Depending on the starting quantum state w.r.t. a given Hermitian operator, we have the mappings: $(Q, \bar{Q}, H) \Longleftrightarrow (d, \delta, \Delta)$ and/or $(\bar{Q}, Q, H) \Longleftrightarrow (d, \delta, \Delta)$.

6 Nilpotent $\mathcal{N} = 2$ SUSY transformations: Supervariable approach and geometrical interpretations

First of all, we focus on the derivation of s_1 within the framework of the supervariable approach. In this regards, we generalize the ordinary variables $(z(t), \bar{z}(t), \psi(t), \bar{\psi}(t))$ to their counterparts supervariables on a $(1, 1)$ -dimensional chiral super-submanifold parametrized by (t, θ) . This super submanifold is a part of the general $(1, 2)$ -dimensional supermanifold on which our theory is generalized. Thus, chiral supervariables and their expansions are:

$$\begin{aligned} z(t) &\longrightarrow Z(t, \theta) = z(t) + \theta f_1(t), \\ \bar{z}(t) &\longrightarrow \bar{Z}(t, \theta) = \bar{z}(t) + \theta f_2(t), \\ \psi(t) &\longrightarrow \Psi(t, \theta) = \psi(t) + i\theta b_1(t), \\ \bar{\psi}(t) &\longrightarrow \bar{\Psi}(t, \theta) = \bar{\psi}(t) + i\theta b_2(t), \end{aligned} \quad (29)$$

where secondary variables $(f_1(t), f_2(t))$ and $(b_1(t), b_2(t))$ are fermionic and bosonic in nature as is evident from the fermionic nature (i.e. $Q^2 = 0$) of Grassmannian variable θ .

To determine these secondary variables in terms of the basic variables, we have to invoke SUSY invariant restrictions (SUSYIRs). In this connection, we note that the following useful quantities remain invariant under s_1 , namely;

$$s_1(\psi) = 0, \quad s_1(\bar{z}) = 0, \quad s_1(z^T \cdot \psi) = 0, \quad s_1[2D_t \bar{z} \cdot z + i\bar{\psi} \cdot \psi] = 0, \quad (30)$$

where $z^T(t) \cdot \psi(t) = z_1 \psi_1 + z_2 \psi_2$. The SUSYIRs demand that all the quantities that are present in the above brackets should remain independent of the “soul” coordinate^{††} θ when they are generalized onto the $(1, 1)$ -dimensional chiral super submanifold. In other words,

^{††}In the old literature (see, e.g. [27]), the Grassmannian variables have been christened as the “soul” coordinates and the ordinary spacetime variables have been called as the “body” coordinates.

we have the following equalities as the SUSYIRs, namely;

$$\begin{aligned} \Psi(t, \theta) = \psi(t) &\implies b_1(t) = 0, & \bar{Z}(t, \theta) = \bar{z}(t) &\implies f_2(t) = 0, \\ Z^T(t, \theta) \cdot \Psi(t, \theta) &= z^T(t) \cdot \psi(t), \\ 2 D_t \bar{Z}(t, \theta) \cdot Z(t, \theta) + i \bar{\Psi}(t, \theta) \cdot \Psi(t, \theta) &= 2 D_t \bar{z}(t) \cdot z(t) + i \bar{\psi}(t) \cdot \psi(t). \end{aligned} \quad (31)$$

From the relationship number three, we obtain the relationship that $f_1(t) \propto \psi(t)$. In this regards, we make a choice and take $f_1(t) = \psi(t)/\sqrt{2}$. This choice, entails upon $b_2(t) = 2 D_t \bar{z}(t)/\sqrt{2}$. Plugging in these values into the expansions, we obtain the following

$$\begin{aligned} Z^{(1)}(t, \theta) &= z(t) + \theta \left(\frac{\psi(t)}{\sqrt{2}} \right) \equiv z(t) + \theta (s_1 z(t)), \\ \bar{Z}^{(1)}(t, \theta) &= \bar{z}(t) + \theta (0) \equiv \bar{z}(t) + \theta (s_1 \bar{z}(t)), \\ \Psi^{(1)}(t, \theta) &= \psi(t) + \theta (0) \equiv \psi(t) + \theta (s_1 \psi(t)), \\ \bar{\Psi}^{(1)}(t, \theta) &= \bar{\psi}(t) + \theta \left(\frac{2 i D_t \bar{z}(t)}{\sqrt{2}} \right) \equiv \bar{\psi}(t) + \theta (s_1 \bar{\psi}(t)), \end{aligned} \quad (32)$$

where the superscript (1) on the supervariables stand for the chiral supervariables that have been obtained after the application of SUSYIRs in (31). A close look at (32) shows that we have already obtained the SUSY transformations s_1 .

The above chiral expansions of the supervariables provide the geometrical meaning to the SUSY transformations s_1 because we note that

$$\frac{\partial}{\partial \theta} \Omega^{(1)}(t, \theta) = s_1 \omega(t) \equiv \pm i [\omega(t), Q]_{\pm}, \quad (33)$$

where $\Omega^{(1)}(t, \theta)$ is the generic chiral supervariable that has been obtained after the application of SUSYIRs in (31) and $\omega(t)$ is the generic ordinary variable. The subscripts (\pm), on the square bracket, denote the existence of (anti)commutator for the given variable $\omega(t)$ being (fermionic)bosonic in nature. Thus, we observe that the operators $(\partial_\theta, s_1, Q)$ are inter-connected. Geometrically, the SUSY transformation s_1 on an ordinary variable is equivalent to the translation of the corresponding supervariable (obtained after SUSYIRs) along the Grassmannian θ -direction of the (1, 1)-dimensional chiral super-submanifold. Furthermore, two successive translations along this Grassmannian direction captures the nilpotency ($s_1^2 = 0, Q^2 = 0$) of s_1 and Q which is nothing other than the nilpotency ($\partial_\theta^2 = 0$) property of the translational generator ∂_θ along the θ -direction.

To derive the SUSY transformations s_2 , we generalize the ordinary variables $(z(t), \bar{z}(t), \psi(t), \bar{\psi}(t))$ onto (1, 1)-dimensional anti-chiral super-submanifold that is characterized by $(t, \bar{\theta})$. In other words, we have the following:

$$\begin{aligned} z(t) &\longrightarrow Z(t, \bar{\theta}) = z(t) + \bar{\theta} f_3(t), \\ \bar{z}(t) &\longrightarrow \bar{Z}(t, \bar{\theta}) = \bar{z}(t) + \bar{\theta} f_4(t), \\ \psi(t) &\longrightarrow \Psi(t, \bar{\theta}) = \psi(t) + i \bar{\theta} b_3(t), \\ \bar{\psi}(t) &\longrightarrow \bar{\Psi}(t, \bar{\theta}) = \bar{\psi}(t) + i \bar{\theta} b_4(t), \end{aligned} \quad (34)$$

where $(f_3(t), f_4(t))$ and $(b_3(t), b_4(t))$ are the pair of fermionic and bosonic secondary variables that have to be determined by exploiting the SUSYIRs on the anti-chiral supervariables. In this connection, it can be checked that the following is true, namely;

$$s_2(\bar{\psi}) = 0, \quad s_2(z) = 0, \quad s_2(\bar{z} \cdot \bar{\psi}^T) = 0, \quad s_2[2\bar{z} \cdot D_t z - i\bar{\psi} \cdot \psi] = 0. \quad (35)$$

Thus, the above invariant quantities would remain independent of the “soul” coordinate $\bar{\theta}$ when they are generalized onto the anti-chiral super sub-manifold. In other words, we have the following SUSYIRs in the form of the equalities, namely;

$$\begin{aligned} Z(t, \bar{\theta}) &= z(t), & \bar{\Psi}(t, \bar{\theta}) &= \bar{\psi}(t), & \bar{Z}(t, \bar{\theta}) \cdot \bar{\Psi}^T(t, \bar{\theta}) &= \bar{z}(t) \cdot \bar{\psi}^T(t), \\ 2\bar{Z}(t, \bar{\theta}) \cdot D_t Z(t, \bar{\theta}) - i\bar{\Psi}(t, \bar{\theta}) \cdot \Psi(t, \bar{\theta}) &= 2\bar{z}(t) \cdot D_t z(t) - i\bar{\psi}(t) \cdot \psi(t), \end{aligned} \quad (36)$$

which lead to the determination of the secondary variables as

$$f_3(t) = 0, \quad b_4(t) = 0, \quad f_4(t) = \frac{\bar{\psi}(t)}{\sqrt{2}}, \quad b_3(t) = \frac{2D_t z(t)}{\sqrt{2}}. \quad (37)$$

The substitution of these secondary variables into the expansions (34) lead to the following

$$\begin{aligned} Z^{(2)}(t, \bar{\theta}) &= z(t) + \bar{\theta}(0) \equiv z(t) + \bar{\theta}(s_2 z), \\ \bar{Z}^{(2)}(t, \bar{\theta}) &= \bar{z}(t) + \bar{\theta} \left(\frac{\bar{\psi}}{\sqrt{2}} \right) \equiv \bar{z}(t) + \bar{\theta}(s_2 \bar{z}), \\ \Psi^{(2)}(t, \bar{\theta}) &= \psi(t) + \bar{\theta} \left(\frac{2iD_t z}{\sqrt{2}} \right) \equiv \psi(t) + \bar{\theta}(s_2 \psi), \\ \bar{\Psi}^{(2)}(t, \bar{\theta}) &= \bar{\psi}(t) + \bar{\theta}(0) \equiv \bar{\psi}(t) + \bar{\theta}(s_2 \bar{\psi}), \end{aligned} \quad (38)$$

where the superscript (2) denotes the expansions of the supervariables after the application of SUSYIRs in (36). The geometrical meaning of the transformations s_2 and its nilpotency can be given in terms of the translational generator $\partial_{\bar{\theta}}$ along $\bar{\theta}$ -direction of the (1, 1)-dimensional anti-chiral super submanifold along exactly similar lines as the ones that have been given for s_1 after equation (32) and (33).

The conserved charges Q and \bar{Q} can also be generalized onto the (anti-)chiral super sub-manifolds where the supervariables would be taken after the application of SUSYIRs.

These expressions in terms of the $\partial_\theta, \partial_{\bar\theta}, d\theta, d\bar\theta$ and (anti-)chiral supervariables are

$$\begin{aligned}
Q &= \frac{\partial}{\partial\theta} \left[2 D_t \bar{Z}^{(1)}(t, \theta) \cdot Z^{(1)}(t, \theta) \right] \equiv \frac{\partial}{\partial\theta} \left[2 D_t \bar{z}(t) \cdot Z^{(1)}(t, \theta) \right], \\
&= \int d\theta \left[2 D_t \bar{Z}^{(1)}(t, \theta) \cdot Z^{(1)}(t, \theta) \right] \equiv \int d\theta \left[2 D_t \bar{z}(t) \cdot Z^{(1)}(t, \theta) \right], \\
Q &= \frac{\partial}{\partial\theta} \left[-i \bar{\Psi}^{(1)}(t, \theta) \cdot \Psi^{(1)}(t, \theta) \right] \equiv \frac{\partial}{\partial\theta} \left[-i \bar{\Psi}^{(1)}(t, \theta) \cdot \psi(t) \right], \\
&= \int d\theta \left[-i \bar{\Psi}^{(1)}(t, \theta) \cdot \Psi^{(1)}(t, \theta) \right] \equiv \int d\theta \left[-i \bar{\Psi}^{(1)}(t, \theta) \cdot \psi(t) \right], \\
\bar{Q} &= \frac{\partial}{\partial\bar\theta} \left[2 \bar{Z}^{(2)}(t, \bar\theta) \cdot D_t Z^{(2)}(t, \bar\theta) \right] \equiv \frac{\partial}{\partial\bar\theta} \left[2 \bar{Z}^{(2)}(t, \bar\theta) \cdot D_t z(t) \right], \\
&= \int d\bar\theta \left[2 \bar{Z}^{(2)}(t, \bar\theta) \cdot D_t Z^{(2)}(t, \bar\theta) \right] \equiv \int d\bar\theta \left[2 \bar{Z}^{(2)}(t, \bar\theta) \cdot D_t z(t) \right], \\
\bar{Q} &= \frac{\partial}{\partial\bar\theta} \left[+i \bar{\Psi}^{(2)}(t, \bar\theta) \cdot \Psi^{(2)}(t, \bar\theta) \right] \equiv \frac{\partial}{\partial\bar\theta} \left[+i \bar{\psi}(t) \cdot \Psi^{(2)}(t, \bar\theta) \right], \\
&= \int d\bar\theta \left[+i \bar{\Psi}^{(2)}(t, \bar\theta) \cdot \Psi^{(2)}(t, \bar\theta) \right] \equiv \int d\bar\theta \left[+i \bar{\psi}(t) \cdot \Psi^{(2)}(t, \bar\theta) \right]. \tag{39}
\end{aligned}$$

The nilpotency of ∂_θ and $\partial_{\bar\theta}$ (i.e. $\partial_\theta^2 = 0, \partial_{\bar\theta}^2 = 0$) implies that the nilpotency of charges Q and \bar{Q} (i.e. $\partial_\theta Q = 0, \partial_{\bar\theta} \bar{Q} = 0$). Furthermore, when we express the above expressions for the charges Q and \bar{Q} in terms of the symmetry transformations s_1 and s_2 and dynamical variables, we observe that

$$Q = s_1 \left(2 D_t \bar{z} \cdot z \right) \equiv s_1 \left(-i \bar{\psi} \cdot \psi \right), \quad \bar{Q} = s_2 \left(2 \bar{z} \cdot D_t z \right) \equiv s_2 \left(+i \bar{\psi} \cdot \psi \right). \tag{40}$$

The nilpotency of charges Q and \bar{Q} can be readily proven by using the constraints $\bar{\psi} \cdot z = 0$ and $\bar{z} \cdot \psi = 0$ that have been taken into considerations in our present endeavor.

We can also capture the invariance of the Lagrangian (1) in terms of the (anti-) chiral supervariables that are obtained after the impositions of the SUSYIRs as given below

$$\begin{aligned}
L \rightarrow \tilde{L}^{(ac)} &= 2 D_t \bar{Z}^{(2)} \cdot D_t Z^{(2)} + \frac{i}{2} \left[\bar{\Psi}^{(2)} \cdot D_t \Psi^{(2)} - D_t \bar{\Psi}^{(2)} \cdot \Psi^{(2)} \right] - 2 g a, \\
L \rightarrow \tilde{L}^{(c)} &= 2 D_t \bar{Z}^{(1)} \cdot D_t Z^{(1)} + \frac{i}{2} \left[\bar{\Psi}^{(1)} \cdot D_t \Psi^{(1)} - D_t \bar{\Psi}^{(1)} \cdot \Psi^{(1)} \right] - 2 g a, \tag{41}
\end{aligned}$$

where the superscripts (c) and (ac) denote the chiral and anti-chiral nature of the Lagrangians $\tilde{L}_0^{(c)}$ and $\tilde{L}_0^{(ac)}$, respectively. The superscripts (1) and (2) on the supervariables correspond to the expansions (32) and (38). Mathematically, we observe that

$$\begin{aligned}
\frac{\partial}{\partial\theta} \left[\tilde{L}^{(c)} \right] &= \frac{d}{dt} \left(\frac{D_t \bar{z} \cdot \psi}{\sqrt{2}} \right) \equiv s_1 L, \\
\frac{\partial}{\partial\bar\theta} \left[\tilde{L}^{(ac)} \right] &= \frac{d}{dt} \left(\frac{\bar{\psi} \cdot D_t z}{\sqrt{2}} \right) \equiv s_2 L. \tag{42}
\end{aligned}$$

The relation (42) provides the geometrical interpretation for the SUSY invariances of the Lagrangian (1). This can be stated that in the following manner. The (anti-)chiral super

Lagrangians (41) are the sum of composite supervariables that have been obtained after the application of the SUSYIRs. The translations of these super Lagrangians along the $\bar{\theta}$ and θ -directions of the (anti-)chiral super-submanifolds are such that they produce the ordinary total time derivatives [cf. (42)] in the ordinary space thereby leaving the action integral ($S = \int dt L$) invariant as the physical variables vanish off at $t = \pm\infty$ on physical grounds.

7 Conclusions

The central theme of our present investigation has been to establish that the *interacting* $\mathcal{N} = 2$ SUSY QMM of the motion of an electron on a sphere, in the background of a magnetic monopole, provides a tractable SUSY model for the Hodge theory. We have accomplished this goal in our present endeavor because we have shown that the discrete and continuous symmetries of our present theory are such that they *together* provide the physical realizations of the de Rham cohomological operators of differential geometry. A few of the specific properties of these cohomological operators are *also* captured in the language of the conserved charges of our present theory in the quantum Hilbert space.

Some of the subtle issues of our present model are the observations that the constraints (e.g. $\bar{z} \cdot z = 1, \bar{\psi} \cdot z = 0, \bar{z} \cdot \psi = 0$) and the equations of motion (10) *together* play very important roles in the proof of the conservation laws as well as in the determination of the $sl(1/1)$ algebra from the symmetry principles and conserved charges. In particular, as discussed in Appendix C, the proof of $\{Q, \bar{Q}\} = H$, from the symmetry transformations $s_1 \bar{Q}$ and $s_2 Q$, requires the mathematical beauty and power of the constraints as well as the equations of motion. It is gratifying to observe that the *interacting* version of our earlier work on the *free* $\mathcal{N} = 2$ SUSY QMM [5] also turns out to be the model for the Hodge theory as the symmetries of the theory provide the physical realizations of (d, δ, Δ) .

We have derived the $\mathcal{N} = 2$ SUSY transformations s_1 and s_2 by exploiting the supervariable approach where we have been theoretically compelled to consider *only* the (anti-)chiral supervariables. This is due to the fact that $\mathcal{N} = 2$ SUSY transformations are nilpotent ($s_1^2 = s_2^2 = 0$) but they are *not* absolutely anticommuting (i.e. $s_1 s_2 + s_2 s_1 \neq 0$). Thus, even though our theory is generalized onto a specific $(1, 2)$ -dimensional supermanifold, we have *not* taken the *full* expansions of the supervariables along $(1, \theta, \bar{\theta}, \theta \bar{\theta})$ -directions of the supermanifold (because the full expansions would automatically imply the validity of absolute anticommutativity). We have also provided the geometrical basis for nilpotency and invariance of the Lagrangian within the framework of our supervariable approach [5,18-21].

Besides our present work on the $\mathcal{N} = 2$ SUSY case of the charge-monopole system, there are many interesting works (see, e.g. [31-40]) on this particular system as well as that of the charge-dyon system (and their very beautiful and diverse super-extensions). For instance, it has been shown, in a couple of very interesting papers [31,32], that our present system exhibits a hidden conical dynamics. The elaborate discussions on the integrability properties of this system, its systematic SUSY generalizations, its connection with the reflectionless potentials, etc., have also attracted a great deal of interest amongst theoretical physicists of very different backgrounds (see, e.g. [31-40] for more details). We would like to lay emphasis on the fact that, in our present investigation, we have concentrated *only*

on the algebraic structures of the symmetries and corresponding conserved charges for this system. However, a whole range of other directions remain to be explored and these remain precisely an open set of interesting problems for the future investigations [41].

Our present $\mathcal{N} = 2$ SUSY QMM is *special* in the sense that there are no singularities in our theory mainly because the constraints $\bar{z} \cdot z = 1$, $\bar{\psi} \cdot z = 0$, $\bar{z} \cdot \psi = 0$ (and their time derivatives) *do* play important role in avoiding them. The other good feature of our present model is the observation that it can be generalized to its counterpart $\mathcal{N} = 4$ version [28]. Thus, it would be a very nice idea to look for the physical realizations of the cohomological operators in the case of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ versions of our present model. We speculate that this understanding might be useful for us in the study of $\mathcal{N} = 2, 4, 8$ SUSY gauge theories within the framework of BRST formalism where we can also look for their topological nature. Such speculations are based on our experience in this kind of study in the context of 2D (non-)Abelian gauge theories [11] which have been shown to present a *new* class of topological field theories. These new topological field theories have been shown to possess the Lagrangian density that look like the Witten-type topological field theories. However, their (anti-)BRST and (anti-)co-BRST symmetries are just like the Schwarz-type topological field theories as they do not incorporate the shift symmetries which are the hallmark of a *typical* Witten-type topological field theories. We are presently busy with these theoretical ideas and we shall report about our progress in our future publications.

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Appendix A: On Superspace Formalism

We invoke here some of the essential ingredients of the superspace formalism to clarify some of the expressions and equations that have been used in our main body of the text. In this connection, we define the supercovariant derivatives (D, \bar{D}) for the *interacting* $\mathcal{N} = 2$ SUSY quantum mechanical model as follows (see, e.g. [29])

$$D = \partial_\theta - i \bar{\theta} D_t, \quad \bar{D} = \partial_{\bar{\theta}} - i \theta D_t. \quad (\text{A.1})$$

The above operators lead to the definition as well as expansions of the chiral $(\Phi(t, \theta, \bar{\theta}))$ and anti-chiral $(\bar{\Phi}(t, \theta, \bar{\theta}))$ supervariables as:

$$\begin{aligned} \Phi(t, \theta, \bar{\theta}) &= z(t) + \theta \psi(t) - i \theta \bar{\theta} D_t z(t), & (D_t z = \dot{z} - i a z), \\ \bar{\Phi}(t, \theta, \bar{\theta}) &= \bar{z}(t) - \bar{\theta} \bar{\psi}(t) + i \theta \bar{\theta} D_t \bar{z}(t), & (D_t \bar{z} = \dot{\bar{z}} + i a \bar{z}), \end{aligned} \quad (\text{A.2})$$

which satisfy $\bar{D} \Phi(t, \theta, \bar{\theta}) = 0$ and $D \bar{\Phi}(t, \theta, \bar{\theta}) = 0$. In the $CP^{(1)}$ model approach, we have the following supersymmetric constraint in terms of $\Phi(t, \theta, \bar{\theta})$ and $\bar{\Phi}(t, \theta, \bar{\theta})$, namely;

$$\bar{\Phi}(t, \theta, \bar{\theta}) \cdot \Phi(t, \theta, \bar{\theta}) - 1 = 0, \quad (\text{A.3})$$

for the description of the motion of an electron on a sphere ($\bar{z} \cdot z = 1$) under background of a monopole. Setting the coefficients of θ , $\bar{\theta}$, $\theta \bar{\theta}$ and constant term equal to zero in (A.3),

we obtain the following expressions for the constraints and a , namely;

$$\bar{z} \cdot z = 1, \quad \bar{z} \cdot \psi = 0, \quad \bar{\psi} \cdot z = 0, \quad a = -\frac{[i(\bar{z} \cdot \dot{z} - \dot{\bar{z}} \cdot z) + (\bar{\psi} \cdot \psi)]}{2(\bar{z} \cdot z)}, \quad (A.4)$$

which have been taken into account in our present endeavor. We have $\mathcal{N} = 2$ SUSY generators for our $\mathcal{N} = 2$ *interacting* quantum mechanical model as:

$$Q = \frac{1}{\sqrt{2}} (\partial_\theta + i \bar{\theta} D_t), \quad \bar{Q} = \frac{1}{\sqrt{2}} (\partial_{\bar{\theta}} + i \theta D_t). \quad (A.5)$$

Together, the above operators satisfy one of the simplest $\mathcal{N} = 2$ SUSY algebra, as

$$\{Q, Q\} \equiv Q^2 = 0, \quad \{\bar{Q}, \bar{Q}\} \equiv \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\} = i D_t. \quad (A.6)$$

The last entry shows that two successive $\mathcal{N} = 2$ SUSY transformations (corresponding to s_1 and s_2) on a variable leads to the covariant “time” translation on the same variable which has been demonstrated to be true in our Sec. 2. In fact, from equation (7), it is clear that the bosonic symmetry ($s_\omega = \{s_1 s_2\}$) is such that it transforms the dynamical variable of our present theory to their “covariant” time derivative modulo a factor of i .

Appendix B: Conservation law for the charges

In this Appendix, we perform some explicit computations that are connected with the proof of the conservations of charges Q and \bar{Q} . In this connection, first of all, we take the straightforward time derivative on the charge Q as follows

$$\frac{dQ}{dt} = \frac{1}{\sqrt{2}} \left[\frac{d\Pi_z}{dt} \cdot \psi + \Pi_z \cdot \frac{d\psi}{dt} \right]. \quad (B.1)$$

Using the equations of motion from (10), we obtain the following expressions for \dot{Q} :

$$\dot{Q} = \frac{1}{\sqrt{2}} \left[-i \left\{ 2a D_t \bar{z} + \frac{1}{2} (\bar{\psi} \cdot \psi + 2g) \dot{\bar{z}} \right\} \cdot \psi + \left\{ 2 D_t \bar{z} + \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) \bar{z} \right\} \cdot \dot{\psi} \right]. \quad (B.2)$$

At this stage, we use the equation of motion w.r.t. $\bar{\psi}$ variable which leads to

$$\dot{\psi} = \frac{i}{2} (\bar{\psi} \cdot \psi + 2g) \psi + i a \psi. \quad (B.3)$$

Multiplication from the left by \bar{z} in the above equation demonstrate that $\bar{z} \cdot \dot{\psi} = 0$ due to the constraint equation $\bar{z} \cdot \psi = 0$. This, in turn, implies that $\dot{\bar{z}} \cdot \psi = 0$ (due to the observation that $d/dt(\bar{z} \cdot \psi) = 0 \Rightarrow \bar{z} \cdot \dot{\psi} + \dot{\bar{z}} \cdot \psi = 0$). It is now trivial to prove that $\dot{Q} = 0$. We lay emphasis on the fact that it is an elegant combination of the equations of motion (10) and the constraints ($\bar{z} \cdot z = 1$, $\bar{z} \cdot \psi = 0$, $\bar{\psi} \cdot z = 0$) which are to be invoked for the proof of $\dot{Q} = 0$. In exactly similar fashion, it can be seen that $\dot{\bar{Q}} = 0$ due to the equations of motion (10) and the constraints $\bar{\psi} \cdot z = 0$, $\dot{\bar{\psi}} \cdot z = 0$, $\bar{\psi} \cdot \dot{z} = 0$. The proof of $\dot{H} \equiv \dot{Q}_\omega = 0$

is elementary at the *classical* as well as the *quantum* levels because the Poisson bracket and/or commutator of H with itself is zero.

Appendix C: Symmetries and Algebraic Structure

We exploit here the ideas of continuous symmetries and their generators to derive one of the simplest form of $sl(1/1)$ algebra that is satisfied amongst the conserved charges (Q, \bar{Q}, Q_ω) of our theory. By exploiting the concept of generators, it is trivial to note that

$$s_1 Q = i \{Q, Q\} = 0, \quad s_2 \bar{Q} = i \{\bar{Q}, \bar{Q}\} = 0, \quad (C.1)$$

because when we compute the l.h.s. of the above equation by exploiting the expression for (Q, \bar{Q}) from (8) and symmetry transformations from (3), we observe that

$$s_1 Q = -\frac{1}{2} (D_t \bar{z} \cdot \psi)(\bar{z} \cdot \psi), \quad s_2 \bar{Q} = \frac{1}{2} (\bar{\psi} \cdot D_t z)(\bar{\psi} \cdot z), \quad (C.2)$$

which turn out to be *zero* on the constrained surface defined by the constraint conditions $\bar{z} \cdot \psi = 0$ and $\bar{\psi} \cdot z = 0$, respectively. In similar fashion, we compute the l.h.s. of the following relationships:

$$s_1 \bar{Q} = i \{\bar{Q}, Q\} = i H, \quad s_2 Q = i \{Q, \bar{Q}\} = i H, \quad (C.3)$$

by exploiting the inputs available in (8) and (3) to obtain the following explicit expressions:

$$\begin{aligned} s_2 Q &= 2i D_t \bar{z} \cdot D_t z + D_t \bar{\psi} \cdot \psi + \frac{1}{2} (\bar{\psi} \cdot D_t z)(\bar{z} \cdot \psi) \\ &\quad + \frac{i}{4} (\bar{\psi} \cdot \psi + 2g) (\bar{\psi} \cdot \psi) - \frac{1}{2} (\bar{\psi} \cdot \psi + 2g) (\bar{z} \cdot D_t z), \\ s_1 \bar{Q} &= 2i D_t \bar{z} \cdot D_t z - (\bar{\psi} \cdot D_t \psi) - \frac{1}{2} (D_t \bar{z} \cdot \psi)(\bar{\psi} \cdot z) \\ &\quad + \frac{i}{4} (\bar{\psi} \cdot \psi + 2g) (\bar{\psi} \cdot \psi) + \frac{1}{2} (D_t \bar{z} \cdot z) (\bar{\psi} \cdot \psi + 2g). \end{aligned} \quad (C.4)$$

Now using the definitions of $D_t z$, $D_t \bar{z}$ and constraints $\bar{\psi} \cdot z = 0$, $\bar{z} \cdot \psi = 0$ plus the equations of motion for ψ and $\bar{\psi}$ from (10), we obtain the following:

$$\begin{aligned} s_1 \bar{Q} &= 2i D_t \bar{z} \cdot D_t z + \frac{1}{2} (\dot{\bar{z}} \cdot z + i a \bar{z} \cdot z) (\bar{\psi} \cdot \psi + 2g) - \frac{i}{4} (\bar{\psi} \cdot \psi + 2g) (\bar{\psi} \cdot \psi), \\ s_2 Q &= 2i D_t \bar{z} \cdot D_t z - \frac{1}{2} (\bar{\psi} \cdot \psi + 2g) (\bar{z} \cdot \dot{z} - i a \bar{z} \cdot z) - \frac{i}{4} (\bar{\psi} \cdot \psi + 2g) (\bar{\psi} \cdot \psi). \end{aligned} \quad (C.5)$$

We further exploit the definition of a and constraint $\bar{z} \cdot z = 1$ and $\frac{d}{dt} (\bar{z} \cdot z - 1) = 0$, in the above equation, to obtain the following

$$s_1 \bar{Q} = i \left[2 D_t \bar{z} \cdot D_t z - \frac{1}{2} (\bar{\psi} \cdot \psi + 2g) (\bar{\psi} \cdot \psi) \right] \equiv i H,$$

$$s_2 Q = i \left[2 D_t \bar{z} \cdot D_t z - \frac{1}{2} (\bar{\psi} \cdot \psi + 2g) (\bar{\psi} \cdot \psi) \right] \equiv iH. \quad (C.6)$$

To be specific, in the computation of $s_1 \bar{Q} = i \{ \bar{Q}, Q \} = iH$, we have used the constraint conditions $\bar{\psi} \cdot z = 0$, $\bar{z} \cdot z = 1$, $\frac{d}{dt}(\bar{z} \cdot z - 1) = 0$ and the definitions of $a, D_t \psi, D_t \bar{z}$. On the other hand, in the explicit composition of $s_2 Q = i \{ Q, \bar{Q} \} = iH$, we have exploited the constraint conditions $\bar{z} \cdot \psi = 0$, $\bar{z} \cdot z = 1$, $\frac{d}{dt}(\bar{z} \cdot z - 1) = 0$ and the definitions of $a, D_t \bar{\psi}, D_t z$. Thus, we note that the constraints as well as the equations of motion are to be exploited judiciously to prove that $s_1 \bar{Q} = s_2 Q = iH$ which, ultimately, implies that $\{Q, \bar{Q}\} = H$.

Appendix D: On Discrete Symmetries

In addition to the discrete symmetry transformations (11), we have the following useful discrete symmetry transformations for the Lagrangian (1), namely;

$$\begin{aligned} z &\rightarrow \pm i \bar{z}, & \bar{z} &\rightarrow \mp i z, & \psi &\rightarrow \pm i \bar{\psi}, & \bar{\psi} &\rightarrow \pm i \psi, \\ t &\rightarrow -t, & a &\rightarrow +a, & g &\rightarrow g. \end{aligned} \quad (D.1)$$

The above symmetry transformations obey all the conditions that have been satisfied by (11). Thus, these symmetries are as good as our transformations in (11). In fact, it can be readily checked that $*L = L, *H = H, *Q = -\bar{Q}, *\bar{Q} = Q$ are true under (D.1).

We dwell a bit on a couple of discrete symmetry transformations for the Lagrangian (1) that are *not* acceptable to us because they do not comply with the strictures laid down by the duality invariant theories [30]. These transformations, for instance, are

$$\begin{aligned} z &\rightarrow \pm \bar{z}, & \bar{z} &\rightarrow \pm z, & \psi &\rightarrow \pm \bar{\psi}, & \bar{\psi} &\rightarrow \pm \psi, \\ t &\rightarrow +t, & a &\rightarrow -a, & g &\rightarrow -g, \end{aligned} \quad (D.2)$$

which leave the Lagrangian (1) invariant. It can be checked that

$$*(*z) = z, \quad *(*\bar{z}) = \bar{z}, \quad *(*\psi) = \psi, \quad *(*\bar{\psi}) = \bar{\psi}. \quad (D.3)$$

With the above observations, we can verify that the following is true:

$$s_2 \Phi = + * s_1 * \Phi, \quad \Phi = z, \bar{z}, \psi, \bar{\psi}. \quad (D.4)$$

However, we note that the reciprocal relation

$$s_1 \Phi = - * s_2 * \Phi, \quad \Phi = z, \bar{z}, \psi, \bar{\psi}, \quad (D.5)$$

is *not* satisfied at all. Thus, the discrete symmetry transformations (D.2) of the Lagrangian (1) are *not* acceptable because they do not comply with the conditions (e.g. reciprocal relationship) laid down by the duality invariant theories [30].

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